Curious behaviour of the diffusion coefficient and friction force for the strongly inhomogeneous HMF model

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Abstract. We present first elements of kinetic theory appropriate to the inhomogeneous phase of the Hamiltonian Mean Field (HMF) model. In particular, we investigate the case of strongly inhomogeneous distributions for $T \to 0$ and exhibit curious behaviour of the force auto-correlation function and friction coefficient. The temporal correlation function of the force has an oscillatory behaviour which averages to zero over a period. By contrast, the effects of friction accumulate with time and the friction coefficient does not satisfy the Einstein relation. On the contrary, it presents the peculiarity to increase linearly with time. Motivated by this result, we provide analytical solutions of a simplified kinetic equation with a time dependent friction. Analogies with self-gravitating systems and other systems with long-range interactions are also mentioned.

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1 Introduction

Recently, there was a renewed interest for the statistical mechanics of systems with long-range interactions [1]. This concerns self-gravitating systems (galaxies) in astrophysics [2,3], large-scale coherent structures (jets and vortices) in geophysical flows [3], bacterial populations (chemotaxis) in biology [4,5], clusters in the Hamiltonian Mean Field (HMF) and Brownian Mean Field (BMF) models [6,7], galactic bars [8], neutral and non-neutral plasmas, dislocation dynamics, planetary formation, cosmology etc. The dynamical evolution of such systems presents a lot of peculiarities [9]. For Hamiltonian systems with long-range interactions, the collisional relaxation time diverges with the number N of particles so that the system experiences two successive types of relaxation: a *collisionless relaxation* on a short timescale of the order of a few dynamical times t_D (called violent relaxation in astrophysics) and a *collisional relaxation* on a long time scale of the order $N^\delta t_D$ with $\delta \geq 1$. The first regime leads to the formation of a metaequilibrium state, or quasi-stationary state (QSS), which is a stable stationary solution of the Vlasov equation that is not necessarily of the Boltzmann form [10–16]. The second regime leads in general to the ordinary statistical equilibrium state described by the Boltzmann distribution. In the case of self-gravitating systems, there may not exist statistical equilibrium and the system can evaporate or collapse (gravothermal catastrophe) [17]. Between the phase of vi-

olent relaxation and the late collisional evolution (equilibrium or collapse) the system passes by a succession of quasi-stationary states which are quasi-stationary solutions of the Vlasov equation slowly evolving with time under the effect of encounters (finite N effects). In astrophysics, this phase is described by the orbit-averaged-Fokker-Planck equation [14,18].

The developement of a kinetic theory for the collisional evolution of systems with long-range interactions is complicated for different reasons. First of all, the temporal correlation function of the force may not decay sufficiently rapidly to vindicate the Markovian approximation that is used in many kinetic theories. For example, in stellar dynamics, the temporal correlation function decreases like t^{-1} leading to a logarithmic divergence of the diffusion coefficient [19]. In the case of the HMF model, the correlation function decreases exponentially rapidly but the correlation time diverges close to the critical point $T \to T_c$ [7,20]. This is a general feature of long-range attractive potentials of interaction [9]. On the other hand, systems with long-range interactions are usually spatially *inhomogeneous* and non-local effects strongly complicate the kinetic theory. In the case of self-gravitating systems, one usually avoids the problem by making a local approximation and developing the kinetic theory as if the system were homogeneous [18]. This is partly justified by the fact that the fluctuations of the gravitational force are dominated by the contribution of the nearest neighbor (the distribution of the force is a particular Lévy law called the Holtzmark distribution) [21]. On the other hand, for the HMF model, the kinetic theory has been developed

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only in the case $T > T_c$ where the system is in a stable homogeneous phase [7,22,23]. In these situations the meanfield force vanishes and, to a first approximation, the particles follow linear trajectories with constant velocity.

One goal of this paper is to present elements of kinetic theories valid for the inhomogeneous phase of the HMF model ($T < T_c$) and show that the situation becomes sensibly different from what we are used to in the case of homogeneous systems. In particular, we shall investigate the case of strongly inhomogeneous systems for $T \to 0$, where the particles cluster around $\theta = 0$. In that case, their mean motion is that of a harmonic oscillator and it is possible to calculate analytically the auto-correlation function of the force and the friction. We find that these quantities present a curious behaviour. The temporal correlation function of the force has an oscillatory evolution which averages to zero over a period. By contrast, the effects of friction accumulate with time and the friction coefficient does not satisfy the Einstein relation. On the contrary, it presents the peculiarity to increase linearly with time. These curious behaviours were previously noted by Kandrup [24] in the case of self-gravitating systems but the consideration of the HMF model allows to obtain more explicit results (devoid of any gravitational divergences) and provides a simple framework where these effects can be studied in detail.

The paper is organized as follows. In Section 2, we recall basic results concerning the structure of the statistical equilibrium states of the HMF model. In Section 3, we develop a kinetic theory of the HMF model valid for both the homogeneous and the inhomogeneous phase. We present a generalized Landau equation describing the evolution of the distribution function of the system as a whole and a generalized Fokker-Planck equation describing the evolution of the distribution function of a test particle (or an ensemble of non-interacting test particles) in a thermal bath of field particles at statistical equilibrium. In Section 4, we calculate the temporal auto-correlation function of the force acting on the test particle. We show that it presents an oscillatory behavior which averages to zero over a period. We also discuss the expression of the diffusion coefficient and its relation to the Kubo formula. In Section 5, we calculate the friction force acting on the test particle. We show that the frictional effects are cumulative and lead to a linear divergence of the friction coefficient for $t \to +\infty$. We also discuss the break-up of the Einstein relation for a strongly inhomogeneous system. In Section 6, we provide the analytical solution of a simplified kinetic equation with a time-dependent friction related to our study.

2 Statistical equilibrium states of the HMF model

The HMF model consists of N particles (of unit mass) moving on a ring and interacting via a cosinusoidal potential. The phase space coordinates (θ_i, v_i) of the particles satisfy the Hamiltonian equations of motion

$$\frac{d\theta_i}{dt} = \frac{\partial H}{\partial v_i}, \qquad \frac{dv_i}{dt} = -\frac{\partial H}{\partial \theta_i},$$

$$H = \sum_{i=1}^{N} \frac{1}{2} v_i^2 - \frac{k}{4\pi} \sum_{i \neq j} \cos(\theta_i - \theta_j). \qquad (1)$$

The HMF model was introduced by several groups (see a short historic in [7]) either as a simple model with long-range interactions mimicking gravitational dynamics [25–27] or as a simplified model for the formation of bars in disk-shape galaxies [8]. The thermodynamic limit corresponds to $N \to +\infty$ in such a way that the rescaled temperature $\eta = kM/4\pi T$ and rescaled energy $\epsilon = 8\pi E/kM^2$ remain of order unity (this can be conveniently accomplished by taking the coupling constant $k \sim 1/N$ right from the begining, in which case $T \sim 1$ and $E \sim N$). In this proper thermodynamic limit, the mean-field approximation becomes exact for $N \to +\infty$, except near the critical point [9]. At statistical equilibrium (see, e.g., [7]), the distribution function can be written $f(\theta, v) = \rho(\theta) f(v)$ with

$$f(v) = \left(\frac{\beta}{2\pi}\right)^{1/2} e^{-\beta \frac{v^2}{2}}, \qquad (2)$$

$$\rho(\theta) = \frac{M}{2\pi I_0(\beta B)} e^{-\beta B \cos \theta}, \qquad (3)$$

where we have adopted the normalization $\int_0^{2\pi} \rho(\theta) d\theta = M$ and $\int_{-\infty}^{+\infty} f(v) dv = 1$ (note that here $M = Nm = N$). Equation (3) is the Boltzmann distribution in a potential $\Phi = B \cos \theta$. The meanfield force experienced by a particle is

$$\langle F \rangle(\theta) = -\Phi'(\theta) = B \sin \theta, \qquad (4)$$

where

$$B = -\frac{k}{2\pi} \int_0^{2\pi} \rho(\theta') \cos \theta' d\theta'. \qquad (5)$$

To obtain equations (4, 5), we have assumed (without loss of generality) that the equilibrium distribution of the system is symmetric with respect to the x-axis. The quantity B is similar to the magnetization in spin systems. It is determined as a function of the temperature (see e.g. [7]) by the implicit equation

$$B = \frac{kM}{2\pi} \frac{I_1(\beta B)}{I_0(\beta B)}. \qquad (6)$$

The energy is given by $E = NT/2 - \pi B^2/k$. For $T > T_c = kM/4\pi$ or $E > E_c = kM^2/8\pi$, the only solution to the above equation is $B = 0$ leading to an homogeneous distribution of particles. For $T < T_c$ or $E < E_c$, the homogeneous phase becomes unstable and a (stable) clustered phase appears with $B \neq 0$. This corresponds to a second order phase transition (see e.g. [7]).

3 The inhomogeneous kinetic equation

We shall discuss here the kinetic theory of the HMF model by using general results coming from the projection operator formalism. This formalism starts from the Liouville equation for the N-body distribution function $P_N(\{\theta_i, v_i\}, t)$ and derives an exact kinetic equation for the one-body distribution function $f(\theta, v, t) = NP_1(\theta, v, t)$ by using projection technics. This equation is then simplified by making some approximations on the correlation function of the field particles. This formalism introduced by Willis & Picard [28] is very general and leads to a form of generalized Landau equation [9]. It was applied by Kandrup [29] in the case of stellar systems, by Chavanis [3,30] for two-dimensional point vortices and in [7] for the HMF model. This formalism is also very close to the linear response theory developed in [24,31] where the friction term (or the drift term in the case of point vortices) is calculated directly from the perturbation of the N-body distribution function of the bath caused by the interaction with the test particle. One interest of this formalism is that it remains valid in the case of non-Markovian and spatially inhomogeneous systems while other formalisms developed in connection with plasma physics (binary collision models, BBGKY hierarchy, diagrammatic expansions, quasilinear theory, ...) usually consider Markovian and homogeneous systems and work in Fourier space. By contrast, the projection operator formalism remains in physical space which enlightens the basic physics. We shall not repeat the intermediate steps of the formalism which can be found in [28–30]. This formalism leads to a general kinetic equation of the form

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \theta} + \langle F \rangle \frac{\partial f}{\partial v} = \frac{\partial}{\partial v} \int_0^t d\tau \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t)$$

$$\times G(t, t - \tau) \left[\mathcal{F}(1 \to 0, t - \tau)\frac{\partial}{\partial v} + \mathcal{F}(0 \to 1, t - \tau)\frac{\partial}{\partial v_1} \right]$$

$$\times f(\theta, v, t - \tau)f(\theta_1, v_1, t - \tau), \qquad (7)$$

where $G(t, t - \tau)$ is the Green function associated to the averaged Liouville operator constructed with the mean-field force $\langle F \rangle(\theta, t)$ and $\mathcal{F}(1 \to 0, t)$ is the fluctuating force created by particle 1 on particle 0 (see, e.g., [29] for more details). Under this form, we clearly see the terms of diffusion and friction (first and second terms in the r.h.s. of Eq. (7)) and their connection to a generalized form of Kubo formula (the time integral of the force autocorrelation function). These points will be developed in the sequel. The ratio of the right hand side (collision term) on the left hand side (meanfield advective term) is of order $1/N$ in the proper thermodynamic limit [7,9] recalled in Section 2. Therefore, for $N \to +\infty$, equation (7) reduces to the Vlasov equation. The collision term takes into account finite N effects and can be viewed as the first order correction to the Vlasov limit in an expansion in N^{-1} (see, e.g., [3] p. 260). Equation (7) can be viewed as a generalization of the Landau equation (initially introduced in

plasma physics) to which it reduces[1] if the system is homogeneous and Markovian [9]. One drawback, however, of the projection operator formalism (or more precisely of the approximations leading to Eq. (7)) is that it ignores collective effects which are important especially close to the critical point. Such collective effects can be taken into account by using the Lenard-Balescu equation in the case of homogeneous systems [7,9,22]. We shall not discuss these collective effects here and shall remain close to the situation considered by Kandrup [29] in the astrophysical setting by adapting and expliciting the calculations in the case of the HMF model.

Equation (7) is an integrodifferential equation (with respect to the variables θ_1, v_1) describing the evolution of the system as a whole. We shall consider here a simpler problem, namely the evolution of a test particle in a bath of field particles with *prescribed* static distribution $f(\theta_1, v_1)$ which is a stable stationary solution of the Vlasov equation. By adapting the projection operator formalism to this situation where $f(\theta_1, v_1)$ is fixed, we find that the time evolution of the density probability $P(\theta, v, t)$ of finding the test particle in (θ, v) at time t is governed by the equation

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial \theta} + \langle F \rangle \frac{\partial P}{\partial v} = \frac{\partial}{\partial v} \int_0^t d\tau \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t)$$

$$\times \left[\mathcal{F}(1 \to 0, t - \tau)\frac{\partial}{\partial v} + \mathcal{F}(0 \to 1, t - \tau)\frac{\partial}{\partial v_1} \right]$$

$$\times P(\theta(t - \tau), v(t - \tau), t - \tau)f(\theta_1(t - \tau), v_1(t - \tau)). \qquad (8)$$

In this paper, we shall consider the evolution of a test particle in a *thermal bath* of field particles at statistical equilibrium described by the distribution (2), (3). In that case, we obtain a kinetic equation of the form

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial \theta} + \langle F \rangle \frac{\partial P}{\partial v} = \frac{\partial}{\partial v} \int_0^t d\tau \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t)$$

$$\times \left\{ \mathcal{F}(1 \to 0, t - \tau)\frac{\partial}{\partial v} - \mathcal{F}(0 \to 1, t - \tau)\beta v_1(t - \tau) \right\}$$

$$\times P(\theta(t - \tau), v(t - \tau), t - \tau)\rho(\theta_1)f(v_1). \qquad (9)$$

The fluctuating force can be written $\mathcal{F}(1 \to 0, t) = F(1 \to 0, t) - \langle F(1 \to 0, t) \rangle$ where

$$F(1 \to 0, t) = -\frac{k}{2\pi}\sin(\theta(t) - \theta_1(t)), \qquad (10)$$

is the exact value of the force created by particle 1 on particle 0 and

$$\langle F(1 \to 0, t) \rangle = \frac{1}{N} \int F(1 \to 0, t)\rho(\theta_1)f(v_1)d\theta_1 dv_1, \qquad (11)$$

is its mean-field value. Equation (9) can be seen as a sort of generalized Fokker-Planck equation. However, the dynamics is generally non-Markovian (see below) so that equation (9) is *not* a Fokker-Planck equation.

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[1] Note, however, that for one-dimensional systems such as the HMF model, the Landau collision term cancels out [7,9,22]; see also last paragraph of [32].

The time integral in equation (9) must be performed by moving the particles with the meanfield force (4) between t and $t - \tau$ (see, e.g., [29]). Accordingly, the quantities $\theta(t-\tau)$ and $v(t-\tau)$ are solutions of the equation of motion

$$\frac{d^2\theta}{dt^2} = B \sin \theta. \tag{12}$$

This is the equation of a nonlinear oscillator. The general solution is given by

$$\int_{\theta_0}^{\theta(t)} \frac{d\phi}{\sqrt{v_0^2 + 2B(\cos\theta_0 - \cos\phi)}} = \pm t, \tag{13}$$

$$v(t) = \pm\sqrt{v_0^2 + 2B(\cos\theta_0 - \cos\theta(t))}. \tag{14}$$

In our previous study [7], we have considered the case $T > T_c$ where the distribution of the bath is homogeneous and the particles follow linear trajectory with constant velocity in a first approximation. In that case, equation (9) reduces to a Fokker-Planck equation of the Kramers form with a friction coefficient given by the Einstein relation. This kinetic equation converges for $t \to +\infty$ to the Maxwellian distribution of the bath [7]. However, the diffusion coefficient decreases rapidly with the velocity (like the Gaussian distribution of the bath) and this leads to anomalous diffusion [22] and to a slow progression of the front in the high velocity tail [23]. Here, we shall consider the case $T \to 0$ where the bath distribution is strongly inhomogeneous. Assuming $B = -\omega^2 < 0$ (the case $B > 0$ is symmetric), the particles cluster around $\theta = 0$ and form a Dirac peak for $T = 0$. For $T \to 0$, they will remain localized around $\theta = 0$. Thus, we can expand equation (12) to first order in θ. It becomes the equation of a harmonic oscillator

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \tag{15}$$

and the equations of motion are explicitly given by

$$\theta(t) = \frac{v_0}{\omega}\sin(\omega t) + \theta_0\cos(\omega t), \tag{16}$$

$$v(t) = v_0\cos(\omega t) - \theta_0\omega\sin(\omega t). \tag{17}$$

For $T \ll T_c$, the parameter B is given by

$$\frac{B}{B_{\max}} = \pm\left(1 - \frac{T}{4T_c}\right) \tag{18}$$

where $B_{\max} = 2T_c = kM/2\pi$ [7]. Therefore, to leading order, we find that the pulsation of the particle trajectory is

$$\omega^2 = \frac{kM}{2\pi}. \tag{19}$$

We shall not attempt in this paper to study the full kinetic equation (9). We shall limit ourselves to analytically study the behaviour of certain quantities that enter in this

equation. To emphasize the physical structure of the kinetic equation (9), we introduce two functions

$$\mathcal{C}(t,\tau) = \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t)\mathcal{F}(1 \to 0, t-\tau)\rho(\theta_1)f(v_1), \tag{20}$$

$$\Psi(t,\tau) = \beta \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t)$$
$$\times \mathcal{F}(1 \to 0, t-\tau)v_1(t-\tau)\rho(\theta_1)f(v_1), \tag{21}$$

in terms of which equation (9) can be rewritten

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial \theta} + \langle F \rangle\frac{\partial P}{\partial v} = \frac{\partial}{\partial v}\int_0^t d\tau\left\{\mathcal{C}(t,\tau)\frac{\partial}{\partial v} \right.$$
$$\left. - \Psi(t,\tau)\right\}P(\theta(t-\tau), v(t-\tau), t-\tau). \tag{22}$$

The function $\mathcal{C}(t,\tau)$ which appears in the *diffusion term* represents the force auto-correlation function. It will be specifically studied in Section 4. On the other hand, the function $\Psi(t,\tau)$ which appears in the *drift term* is connected to the friction force which will be specifically studied in Section 5 (see in particular the quantity I). Then, the difficulty with equation (22) lies in the fact that it is non Markovian: we need to know the value of $P(\theta, v, t)$ at previous times to pass from t to $t + dt$. In Section 6, in order to get some insight into the problem, we shall consider a simpler kinetic equation where we neglect non-Markovian terms altogether but keep the time dependence of the diffusion and friction coefficients.

4 The force auto-correlation function

4.1 The temporal correlation function

One quantity of great interest in the kinetic theory is the force auto-correlation function. Indeed, in ordinary circumstances, the diffusion coefficient in the Fokker-Planck equation is expressed as the time integral of the auto-correlation function through the Kubo formula. The temporal auto-correlation of the fluctuating force can be decomposed into

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = \langle F(0)F(t)\rangle - \langle F(0)\rangle\langle F(t)\rangle, \tag{23}$$

where $\mathcal{F}(t) = F(t) - \langle F(t)\rangle$ is the total fluctuating force acting on the test particle at time t. Using $F(t) = \sum_i F(i \to 0, t)$, we get

$$\langle F(0)F(t)\rangle = \sum_{i,j}\langle F(i \to 0, 0)F(j \to 0, t)\rangle$$
$$= \sum_i\langle F(i \to 0, 0)F(i \to 0, t)\rangle$$
$$+ \sum_{i\neq j}\langle F(i \to 0, 0)F(j \to 0, t)\rangle. \tag{24}$$

Since the N-body distribution of the bath is a product of N one-body distributions (see, e.g., [7]) and since the particles are identical, we obtain

$$\langle F(0)F(t)\rangle = N\langle F(1\to 0,0)F(1\to 0,t)\rangle$$
$$+N(N-1)\langle F(1\to 0,0)\rangle\langle F(1\to 0,t)\rangle. \quad (25)$$

Accounting that $\langle F(t)\rangle = N\langle F(1\to 0,t)\rangle$, we get

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = N\langle F(1\to 0,0)F(1\to 0,t)\rangle$$
$$- N\langle F(1\to 0,0)\rangle\langle F(1\to 0,t)\rangle, \quad (26)$$

which can be written

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = N\langle \mathcal{F}(1\to 0,0)\mathcal{F}(1\to 0,t)\rangle, \quad (27)$$

where $\mathcal{F}(1\to 0,t) = F(1\to 0,t) - \langle F(1\to 0,t)\rangle$ is the fluctuating force produced by particle 1 on the test particle. Explicitly,

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = \int \mathcal{F}(1\to 0,0)\mathcal{F}(1\to 0,t)\rho(\theta_1)f(v_1)d\theta_1 dv_1. \quad (28)$$

We note that this combination of terms enters in the diffusion term in equation (9); this is precisely the function $\mathcal{C}(t,\tau)$ defined in equation (20) where we have taken the origin of times at $t=0$. Let us first compute the quantity

$$\{F(0)F(t)\} = N\langle F(1\to 0,0)F(1\to 0,t)\rangle. \quad (29)$$

Explicitly, we have

$$\{F(0)F(t)\} = \int F(1\to 0,0)F(1\to 0,t)$$
$$\times \rho(\theta_1)f(v_1)d\theta_1 dv_1. \quad (30)$$

Using equation (10), we get

$$\{F(0)F(t)\} = \frac{k^2}{4\pi^2}\int \sin(\theta-\theta_1)\sin(\theta(t)-\theta_1(t))$$
$$\times \rho(\theta_1)f(v_1)d\theta_1 dv_1, \quad (31)$$

where $\theta_i(t)$ denotes the position at time t of the i-th particle located at θ_i at $t=0$. Now, using the equation of motion (16), we obtain

$$\theta(t) - \theta_1(t) = \frac{u}{\omega}\sin(\omega t) + \phi\cos(\omega t), \quad (32)$$

where $\phi = \theta - \theta_1$ and $u = v - v_1$. Substituting these results in equation (31) we get

$$\{F(0)F(t)\} = \frac{k^2}{4\pi^2}\int dv_1 f(v_1)\int d\theta_1\rho(\theta_1)\sin\phi$$
$$\times \sin\left[\frac{u}{\omega}\sin(\omega t) + \phi\cos(\omega t)\right]. \quad (33)$$

In the $T\to 0$ limit, the spatial distribution of the particles (3) can be approximated by

$$\rho(\theta) = M\omega\left(\frac{\beta}{2\pi}\right)^{1/2}e^{-\beta\omega^2\frac{\theta^2}{2}}. \quad (34)$$

Using furthermore $\sin(x)\simeq x$ in equation (33) for small θ, v and performing the Gaussian integrations, we finally obtain

$$\{F(0)F(t)\} = \frac{Mk^2}{4\pi^2}\frac{v\theta}{\omega}\sin(\omega t) + \frac{Mk^2}{4\pi^2}\left(\theta^2 + \frac{1}{\beta\omega^2}\right)\cos(\omega t). \quad (35)$$

In the preceding expansions, we have implicitly assumed that the coordinates of the field particles *and* of the test particle scale as $v\sim\sqrt{2/\beta}$ and $\theta\sim(1/\omega)\sqrt{2/\beta}$. Therefore, our asymptotic expansion is valid to order T for $T\to 0$.

Now, the correlation function of the fluctuating force is given by

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = \{F(0)F(t)\} - N\langle F(1\to 0,0)\rangle\langle F(1\to 0,t)\rangle, \quad (36)$$

where

$$\langle F(1\to 0,t)\rangle = -\frac{k}{2\pi N}\int\sin(\theta(t)-\theta_1(t))\rho(\theta_1)f(v_1)d\theta_1 dv_1. \quad (37)$$

Using the same approximations as before, we obtain

$$\langle F(1\to 0,t)\rangle = -\frac{k}{2\pi}\left[\frac{v}{\omega}\sin(\omega t) + \theta\cos(\omega t)\right]. \quad (38)$$

Combining the previous results, we get

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = \frac{k^2}{4\pi^2}\frac{M}{\beta\omega^2}\cos(\omega t). \quad (39)$$

Using equation (19), we finally obtain

$$\langle \mathcal{F}(0)\mathcal{F}(t)\rangle = \frac{k}{2\pi\beta}\cos(\omega t) + O(T^2). \quad (40)$$

We note that, to order T, the correlation function of the fluctuating force depends only on the ellapsed time t and not on the initial instant $t=0$. We also note that the correlation function is periodic with the same pulsation ω as the particle trajectory and that it averages to zero over a period.

4.2 The diffusion coefficient

If $\Delta v = \int_0^t \mathcal{F}(t')dt'$ denotes the increment of velocity of the test particle caused by the fluctuating force during an interval of time t, we define the diffusion coefficient by

$$D(t) = \frac{\langle(\Delta v)^2\rangle}{2t}. \quad (41)$$

This can be rewritten

$$D(t) = \frac{1}{2t}\int_0^t dt'\int_0^t dt''\langle \mathcal{F}(t')\mathcal{F}(t'')\rangle. \quad (42)$$

Since the correlation function depends only on the time interval $|t'' - t'|$, we also have

$$D(t) = \frac{1}{t} \int_0^t dt' \int_{t'}^t dt'' \langle \mathcal{F}(0)\mathcal{F}(t'' - t') \rangle. \qquad (43)$$

Setting $\tau = t'' - t'$, we get

$$D(t) = \frac{1}{t} \int_0^t dt' \int_0^{t-t'} d\tau \langle \mathcal{F}(0)\mathcal{F}(\tau) \rangle, \qquad (44)$$

or, equivalently,

$$D(t) = \frac{1}{t} \int_0^t d\tau \int_0^{t-\tau} dt' \langle \mathcal{F}(0)\mathcal{F}(\tau) \rangle. \qquad (45)$$

Finally, we obtain

$$D(t) = \frac{1}{t} \int_0^t d\tau (t - \tau) \langle \mathcal{F}(0)\mathcal{F}(\tau) \rangle. \qquad (46)$$

If the correlation function decreases sufficiently rapidly with time, taking $t \to +\infty$, we obtain the Kubo formula

$$D = \int_0^{+\infty} \langle \mathcal{F}(0)\mathcal{F}(\tau) \rangle d\tau. \qquad (47)$$

However, since in the present situation the temporal correlation function has an oscillatory behaviour, this formula is not applicable. According to equation (46), the diffusion coefficient can be written $D = D_1 - D_2$ where

$$D_1 = \int_0^t \langle \mathcal{F}(0)\mathcal{F}(\tau) \rangle d\tau, \qquad (48)$$

and

$$D_2 = \frac{1}{t} \int_0^t \langle \mathcal{F}(0)\mathcal{F}(\tau) \rangle \tau d\tau. \qquad (49)$$

Using equation (39), we find that

$$D_1 = \frac{k^2}{4\pi^2} \frac{M}{\beta\omega^3} \sin(\omega t), \qquad (50)$$

$$D_2 = \frac{k^2}{4\pi^2} \frac{M}{\beta\omega^3} \left[\sin(\omega t) + \frac{1}{\omega t} \cos(\omega t) - \frac{1}{\omega t} \right]. \qquad (51)$$

Thus, we obtain

$$D(t) = \frac{k^2}{4\pi^2} \frac{M}{\beta\omega^3} \frac{1 - \cos(\omega t)}{\omega t}. \qquad (52)$$

The diffusion coefficient is periodic and goes to zero at each time $t_n = (2\pi/\omega)n$ with $n = 1, 2, \ldots$ (see Fig. 1). For $t \to 0$, it behaves like

$$D(t) = \frac{k^2}{8\pi^2} \frac{M}{\beta\omega^2} t = \frac{k}{4\pi\beta} t. \qquad (53)$$

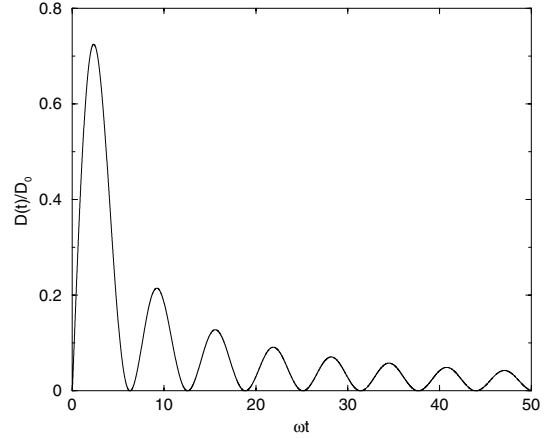On the other hand, $D(t) \to 0$ for $t \to +\infty$.

Fig. 1. The function $D(t)/D_0$ where $D_0 = \frac{k^2}{4\pi^2} \frac{M}{\beta\omega^3}$.

4.3 The spatial correlation function

Let us finally provide the exact expression of the spatial correlation function $\langle \mathcal{F}(\theta)\mathcal{F}(\theta') \rangle$ in the case where the correlations between particles are neglected (as before). The effect of correlations is considered in [7,9] for the homogeneous phase. The case of the inhomogeneous phase will be considered elsewhere.

Repeating the same steps as in Section 4.1, the spatial correlations of the fluctuating force can be written

$$\langle \mathcal{F}(\theta)\mathcal{F}(\theta') \rangle = \{F(\theta)F(\theta')\} - \frac{1}{N} \langle F(\theta) \rangle \langle F(\theta') \rangle, \quad (54)$$

where

$$\{F(\theta)F(\theta')\} = \frac{k^2}{4\pi^2} \int_0^{2\pi} \sin(\theta - \theta_1) \sin(\theta' - \theta_1)\rho(\theta_1)d\theta_1, \qquad (55)$$

and

$$\langle F(\theta) \rangle = B \sin\theta = \frac{kM}{2\pi} \frac{I_1(\beta B)}{I_0(\beta B)} \sin\theta. \qquad (56)$$

Using equation (3), we have

$$\{F(\theta)F(\theta')\} = \frac{k^2 M}{8\pi^3 I_0(\beta B)}$$
$$\times \int_0^{2\pi} \sin(\theta - \theta_1) \sin(\theta' - \theta_1)e^{-\beta B \cos\theta_1}d\theta_1. \quad (57)$$

Expanding the trigonometric functions and using the identities

$$\int_0^{2\pi} \sin\theta_1 \cos\theta_1 e^{-\beta B \cos\theta_1}d\theta_1 = 0, \qquad (58)$$

$$\int_0^{2\pi} \sin^2\theta_1 e^{-\beta B \cos\theta_1}d\theta_1 = \frac{2\pi}{\beta B}I_1(\beta B), \qquad (59)$$

the integrals in equation (57) can be easily performed. Then, using equations (56) and (54), we finally obtain

$$\langle \mathcal{F}(\theta)\mathcal{F}(\theta') \rangle = \frac{k^2 M}{4\pi^2}\left[\frac{I_1(x)}{xI_0(x)}\cos(\theta-\theta') \right.$$
$$\left. + \left(1 - \frac{I_1(x)^2}{I_0(x)^2} - \frac{2I_1(x)}{xI_0(x)}\right)\sin\theta\sin\theta' \right], \quad (60)$$

where we have set $x = \beta B$. We note that, due to the inhomogeneity of the system, the correlation function of the fluctuating force is not a function of $|\theta - \theta'|$ alone.

Let us consider particular cases. If we take $\theta' = \theta$, we see that $\langle \mathcal{F}(\theta)^2 \rangle$ depends on θ through a term $\sin^2\theta$. If we take $\theta = 0$ and $\theta' = \phi$, we obtain

$$\langle \mathcal{F}(0)\mathcal{F}(\phi) \rangle = \frac{k^2 M}{4\pi^2}\frac{I_1(x)}{xI_0(x)}\cos\phi. \quad (61)$$

For $T \geq T_c$ (homogeneous phase), we have $x = 0$ and equation (61) reduces to

$$\langle F(0)F(\phi) \rangle = \frac{k^2 M}{8\pi^2}\cos\phi. \quad (62)$$

We recover the result of [7,9] when the correlations between particles are neglected. For $T < T_c$, using equation (6), we find that

$$\langle \mathcal{F}(0)\mathcal{F}(\phi) \rangle = \frac{kT}{2\pi}\cos\phi. \quad (63)$$

The dependence of this correlation function with the temperature is plotted in Figure 2. Coming back to the function (60), and considering the homogeneous phase $x = 0$, we get

$$\langle \mathcal{F}(\theta)\mathcal{F}(\theta') \rangle = \frac{k^2 M}{4\pi^2}\cos(\theta-\theta'). \quad (64)$$

On the other hand, for $T \to 0$ corresponding to $x \to +\infty$, we find that

$$\langle \mathcal{F}(\theta)\mathcal{F}(\theta') \rangle = \frac{kT}{2\pi}\cos(\theta+\theta') + O(T^2). \quad (65)$$

In particular,

$$\langle \mathcal{F}(\theta)^2 \rangle = \frac{kT}{2\pi}\cos(2\theta) + O(T^2). \quad (66)$$

If we assume that $\theta \sim \sqrt{T}$ as in Section 4.1, then

$$\langle \mathcal{F}^2 \rangle = \frac{kT}{2\pi} + O(T^2), \quad (67)$$

which coincides with equation (40) when $t = 0$.

5 The friction force

We shall now compute the frictional force

$$\langle F_{\mathrm{fr}}(t) \rangle = \beta \int_0^t d\tau \int d\theta_1 dv_1 \mathcal{F}(1 \to 0, t)\mathcal{F}(0 \to 1, t-\tau)$$
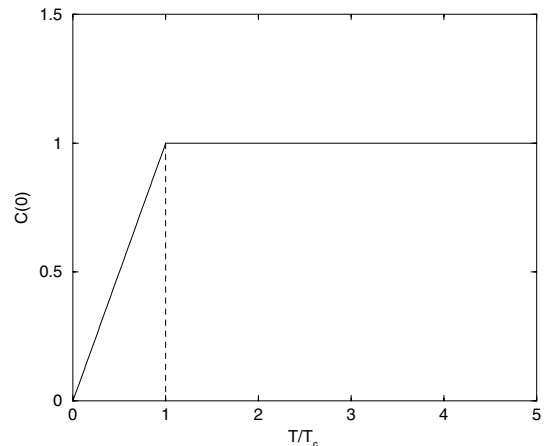$$\times v_1(t-\tau)\rho(\theta_1)f(v_1), \quad (68)$$

Fig. 2. The normalized variance of the fluctuating force $C(0) = \frac{8\pi^2}{k^2 M}\langle \mathcal{F}(0)^2 \rangle$ at $\theta = 0$ as a function of the temperature when the correlations between particles are neglected.

experienced by a test particle with *prescribed* trajectory given by equation (16). The expression (68) can be directly obtained from a linear response theory as done by Kandrup [24] for the gravitational interaction. The friction arises as the response of the field particles to the perturbation caused by the test particle as in a polarization process. Note that when the test particle is described by a distribution function $P(\theta, v, t)$ [instead of having a prescribed trajectory] as in Section 3, the combination of terms (68) also enters in the friction term of equation (9) although the expression is more complicated as it involves the history of the distribution function of the test particle.

Since the fluctuating force averages to zero, equation (68) can also be written as

$$\langle F_{\mathrm{fr}}(t) \rangle = \beta \int_0^t d\tau \int d\theta_1 dv_1 F(1 \to 0, t)\mathcal{F}(0 \to 1, t-\tau)$$
$$\times v_1(t-\tau)\rho(\theta_1)f(v_1). \quad (69)$$

Taking the origin of times at $t = 0$, we shall first compute the quantity

$$I(t) = \int F(1 \to 0, 0)F(1 \to 0, t)v_1(t)$$
$$\times \rho(\theta_1)f(v_1)d\theta_1 dv_1, \quad (70)$$

which is related to the function $\Psi(t, \tau)$ defined in equation (21). Using the fact that

$$v_1(t) = v_1\cos(\omega t) - \theta_1\omega\sin(\omega t), \quad (71)$$

we can write $I = I_1 + I_2$ where

$$I_1(t) = \cos(\omega t)\int F(1 \to 0, 0)F(1 \to 0, t)v_1$$
$$\times \rho(\theta_1)f(v_1)d\theta_1 dv_1, \quad (72)$$

$$I_2(t) = -\omega\sin(\omega t)\int F(1 \to 0, 0)F(1 \to 0, t)\theta_1$$
$$\times \rho(\theta_1)f(v_1)d\theta_1 dv_1. \quad (73)$$

The first integral can be rewritten

$$I_1(t) = \frac{k^2}{4\pi^2} \cos(\omega t) \int dv_1 f(v_1) v_1 \int d\theta_1 \rho(\theta_1) \sin \phi$$
$$\times \sin\left[\frac{u}{\omega} \sin(\omega t) + \phi \cos(\omega t)\right]. \quad (74)$$

Using the same approximations as in Section 4.1, we obtain

$$I_1(t) = -\frac{k^2}{4\pi^2} \frac{M\theta}{2\omega\beta} \sin(2\omega t). \quad (75)$$

Our asymptotic expansion is here valid to order $T^{3/2}$. The second integral can be rewritten

$$I_2(t) = -\frac{k^2}{4\pi^2} \omega \sin(\omega t) \int dv_1 f(v_1) \int d\theta_1 \rho(\theta_1) \theta_1 \sin \phi$$
$$\times \sin\left[\frac{u}{\omega} \sin(\omega t) + \phi \cos(\omega t)\right], \quad (76)$$

leading to

$$I_2(t) = \frac{k^2}{4\pi^2} \frac{Mv}{\beta\omega^2} \sin^2(\omega t) + \frac{k^2}{4\pi^2} \frac{M\theta}{\beta\omega} \sin(2\omega t). \quad (77)$$

Summing these results, we get

$$I(t) = \frac{k^2}{4\pi^2} \frac{Mv}{\beta\omega^2} \sin^2(\omega t) + \frac{k^2}{8\pi^2} \frac{M\theta}{\beta\omega} \sin(2\omega t). \quad (78)$$

Since $\sin^2(\omega t) = [1 - \cos(2\omega t)]/2$, the function $I(t)$ is periodic with pulsation 2ω, i.e. twice the pulsation of the orbiting particles. However, this function also contains a constant component which does not average to zero over a period. We have instead

$$\langle I(t) \rangle = \frac{1}{T} \int_0^T I(t) dt = \frac{k^2 Mv}{8\pi^2 \beta\omega^2}, \quad (79)$$

where T here denotes the period. This implies that *the effects of the friction accumulate with time.*

To obtain the complete friction force, we also have to calculate the quantity

$$J(t) = \int F(1 \to 0, 0) \langle F(1 \to 0, t) \rangle v_1(t)$$
$$\times \rho(\theta_1) f(v_1) d\theta_1 dv_1. \quad (80)$$

Using

$$\langle F(0 \to 1, t) \rangle = -\frac{k}{2\pi} \left[\frac{v_1}{\omega} \sin(\omega t) + \theta_1 \cos(\omega t)\right], \quad (81)$$

and performing the same approximations as before, we find that

$$J(t) = \frac{k^2}{4\pi^2} \int \phi\left(\frac{v_1}{\omega} \sin\omega t + \theta_1 \cos\omega t\right)$$
$$\times (v_1 \cos\omega t - \theta_1\omega \sin\omega t) \rho(\theta_1) f(v_1) d\theta_1 dv_1. \quad (82)$$

By using the parity of the velocity and angular distributions, we get

$$J(t) = \frac{k^2\theta}{8\pi^2} \sin(2\omega t) \int \left(\frac{v_1^2}{\omega} - \theta_1^2\omega\right) \rho(\theta_1) f(v_1) d\theta_1 dv_1. \quad (83)$$

Finally, performing the Gaussian integrations, we find that $J = 0$. Therefore, the frictional force at time t is simply given by

$$\langle F_{\text{fr}}(t) \rangle = -\beta \int_0^t I(-\tau) d\tau. \quad (84)$$

Focusing on the component proportional to the velocity of the test particle (which contains the diverging contribution for $t \to +\infty$), we get

$$\langle F_{\text{fr}}(t) \rangle = -\frac{k^2 M}{4\pi^2} \frac{v(t)}{\omega^2} \int_0^t \sin^2(\omega\tau) d\tau \equiv -\xi(t) v. \quad (85)$$

We note that, in the present situation, the friction coefficient is *not* given by an Einstein relation. Keeping only the diverging contribution coming from the non-vanishing averaged value of $I(t)$ given by equation (79), we find that

$$\xi(t) = \frac{k^2 M}{8\pi^2} \frac{1}{\omega^2} t = \frac{kt}{4\pi}, \quad (86)$$

so that

$$\langle F_{\text{fr}}(t) \rangle = -\frac{kt}{4\pi} v(t). \quad (87)$$

Therefore, the friction coefficient *increases linearly with time.* If we consider only that part (linear in t) of the friction coefficient, we find that it is related to the diffusion coefficient (52) by

$$\xi(t) = \beta D(t) \frac{\omega^2 t^2}{2(1 - \cos\omega t)}, \quad (88)$$

which resembles the Einstein relation. For $t \to 0$, we find that $\xi(t) = D(t)\beta$.

If we now consider the full expression of the friction force, we get

$$\langle F_{\text{fr}}(t) \rangle = -\frac{k^2 M}{8\pi^2} \frac{1}{\omega^2} \left[v\left(t - \frac{\sin 2\omega t}{2\omega}\right) + \frac{\theta}{2}(\cos 2\omega t - 1)\right]. \quad (89)$$

The temporal behaviours of the components of the friction force proportional to v and θ are represented in Figure 3. For $t \to +\infty$, we recover equation (87) and for $t \to 0$, we have

$$\langle F_{\text{fr}}(t) \rangle = -\frac{k^2 M}{4\pi^2} \left(\frac{v}{3} t^3 - \frac{\theta}{2} t^2\right). \quad (90)$$

Finally, with the quantities defined in Sections 4 and 5, the kinetic equation (9) can be written

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial \theta} + \langle F \rangle \frac{\partial P}{\partial v} =$$
$$\frac{\partial}{\partial v} \int_0^t d\tau \left\{\langle \mathcal{F}(t)\mathcal{F}(t - \tau) \rangle \frac{\partial}{\partial v} + \beta I(-\tau)\right\}$$
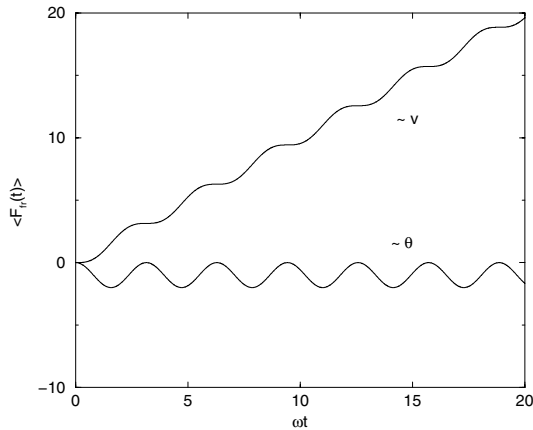$$\times P(\theta(t - \tau), v(t - \tau), t - \tau). \quad (91)$$

Fig. 3. The temporal behaviour of the components of the friction force proportional to v and θ.

Using equation (40) and keeping only the constant term in equation (78), we get

$$\frac{\partial P}{\partial t} + v\frac{\partial P}{\partial \theta} + \langle F\rangle\frac{\partial P}{\partial v} = \frac{k^2 M}{8\pi^2}\frac{1}{\beta\omega^2}\frac{\partial}{\partial v}$$
$$\times \int_0^t d\tau\left\{2\cos(\omega\tau)\frac{\partial}{\partial v} + \beta v(t)\right\}P(\theta(t-\tau), v(t-\tau), t-\tau).$$
(92)

The study of these kinetic equations is beyond the scope of the present paper. Let us simply state that the kinetic theory of inhomogeneous systems with long-range interactions is far from being completely understood. In particular, it is not clear whether equation (92), and more generally equation (7), satisfies an H-theorem. Indeed, starting from the general kinetic equation (7), introducing the Boltzmann entropy $S = -\int f\ln f d\theta dv$ and using standard methods, we can put the rate of entropy production in the form

$$\dot{S} = \frac{1}{2}\int d\theta dv d\theta_1 dv_1 \frac{1}{ff_1}\int_0^t d\tau Q(t)G(t, t-\tau)Q(t-\tau),$$
(93)

where

$$Q(t) \equiv \left[\mathcal{F}(1\to 0, t)\frac{\partial}{\partial v} + \mathcal{F}(0\to 1, t)\frac{\partial}{\partial v_1}\right]$$
$$\times f(\theta, v, t)f(\theta_1, v_1, t). \quad (94)$$

In the course of the calculations, we have inverted the dummy variables θ, v and θ_1, v_1 and taken the half sum of the resulting expressions (see, e.g., [33]). We see that the H-theorem $\dot{S} \geq 0$ is not granted. This depends on the temporal correlations of $Q(t)$. Furthermore, it is not clear whether equation (7) conserves energy and whether it converges towards some form of equilibrium (Maxwellian or other) for large times. If we take for granted that the system *must* converge towards statistical equilibrium for $t \to +\infty$, this implies that the kinetic theory may not

be complete; one may have to relax certain simplifying approximations and consider next order terms in the expansion in $1/N$ of the correlation functions. Alternatively, if we rely on the kinetic theory to give the justification (or not) of the statistical equilibrium state, one may conclude that the Boltzmannian distribution may not be reached (at least in a strict sense) for inhomogeneous systems. There might be dynamical anomalies preventing the system from reaching equilibrium. These are open questions left for future works. Note, however, that the preceding discussion does not favour other forms of entropy. The Boltzmann entropy remains the most fundamental even if the H-theorem may not be rigorously valid.

6 Kinetic equation with time dependent friction

6.1 General solution

The kinetic equation (92) is complicated due to non Markovian effects and spatial inhomogeneity encapsulated in the advection term. In addition, one striking novel aspect of our study is the time dependence of the diffusion coefficient and friction force. Therefore, in a first attempt to investigate the effects produced by such terms, we shall consider a simpler kinetic equation where we neglect non-Markovian effects and advective terms altogether but keep the time dependence of the diffusion and friction coefficients. In that case, equation (92) reduces to

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left(D(t)\frac{\partial P}{\partial v} + \xi(t)Pv\right), \quad (95)$$

where $D(t)$ and $\xi(t)$ are given by equations (48) and (86). More generally, we shall consider equations of the form (95) for arbitrary functions of time $D(t)$ and $\xi(t)$. An interesting aspect of the problem is that such equations can be solved analytically. We note that when D and ξ are constant, we get the familiar Kramers equation. Its stationary solution is the Maxwellian $P_e = Ae^{-\beta v^2/2}$ provided that D and ξ are related to each other by the Einstein relation $\xi = D\beta$. We shall now consider the case where D and ξ depend on time. By redefining time such that $dt' = D(t)dt$, we are led to consider, without loss of generality, equations of the form

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left(D\frac{\partial P}{\partial v} + \gamma h(t)Pv\right), \quad (96)$$

where D and γ are constant parameters.

Taking the Fourier transform of equation (96) with the conventions

$$P(v) = \int \hat{P}(\xi)e^{i\xi v}d\xi, \qquad \hat{P}(\xi) = \int P(v)e^{-i\xi v}\frac{dv}{2\pi}, \quad (97)$$

and using the relation

$$P(v)v = \int v\hat{P}(\xi)e^{i\xi v}d\xi$$

$$= \frac{1}{i}\int \hat{P}(\xi)\frac{\partial}{\partial\xi}(e^{i\xi v})d\xi = -\frac{1}{i}\int \frac{\partial\hat{P}}{\partial\xi}e^{i\xi v}d\xi, \quad (98)$$

we get

$$\frac{\partial\hat{P}}{\partial t} = -D\xi^2\hat{P} - \gamma h(t)\xi\frac{\partial\hat{P}}{\partial\xi}. \quad (99)$$

We introduce the change of variables

$$f(y,t) = \hat{P}(H_1(t)y,t), \qquad \xi = H_1(t)y \quad (100)$$

and choose the function $H_1(t)$ such that

$$\frac{\dot{H}_1}{H_1} = \gamma h(t). \quad (101)$$

Substituting equation (100) in equation (99), we find that $f(y,t)$ satisfies

$$\frac{\partial f}{\partial t} + DH_1^2(t)y^2 f = 0. \quad (102)$$

Let $H(t)$ be the primitive of $h(t)$ such that

$$H(t) = \int_0^t h(\tau)d\tau. \quad (103)$$

Then, we choose H_1, solution of equation (101), such that

$$H_1(t) = e^{\gamma H(t)}. \quad (104)$$

By convention, $H(0) = 0$ and $H_1(0) = 1$. Equation (102) can be integrated leading to

$$f(y,t) = f(y,0)e^{-H_2(t)y^2}, \quad (105)$$

where we have defined

$$H_2(t) = D\int_0^t H_1(\tau)^2 d\tau. \quad (106)$$

Returning to original variables, we obtain

$$\hat{P}(\xi,t) = \hat{P}_0\left(\frac{\xi}{H_1(t)}\right)e^{-\chi^2(t)\xi^2}, \quad (107)$$

where we have defined

$$\chi^2(t) \equiv \frac{H_2(t)}{H_1(t)^2} = D\int_0^t e^{2\gamma[H(\tau)-H(t)]}d\tau. \quad (108)$$

Defining

$$q(v) = P_0(H_1(t)v) \quad \leftrightarrow \quad \hat{q}(\xi) = \frac{1}{H_1(t)}\hat{P}_0\left(\frac{\xi}{H_1(t)}\right) \quad (109)$$

$$g(v) = G(v/2\chi(t)) \quad \leftrightarrow \quad \hat{g}(\xi) = 2\chi(t)\hat{G}(2\chi(t)\xi) \quad (110)$$

where

$$G(z) = e^{-z^2} \quad \leftrightarrow \quad \hat{G}(\xi) = \frac{1}{2\sqrt{\pi}}e^{-\xi^2/4} \quad (111)$$

we can rewrite equation (107) in the form

$$\hat{P}(\xi,t) = \sqrt{\pi}\frac{H_1(t)}{\chi(t)}\hat{q}(\xi)\hat{g}(\xi). \quad (112)$$

Taking the inverse Fourier transform, we can express the solution of equation (96) as a convolution

$$P(v,t) = \sqrt{\pi}\frac{H_1(t)}{\chi(t)}\int q(v-v')g(v')\frac{dv'}{2\pi}, \quad (113)$$

or, equivalently

$$P(v,t) = \frac{H_1(t)}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}P_0[H_1(t)(v-2\chi(t)x)]dx. \quad (114)$$

By direct substitution, we can check that equation (114) is indeed solution of equation (96). For $\gamma = 0$ (pure diffusion) we find that

$$P(v,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-x^2}P_0(v-2x\sqrt{Dt})dx. \quad (115)$$

For $D = 0$, equation (114) reduces to

$$P(v,t) = H_1(t)P_0[H_1(t)v]. \quad (116)$$

This can also be obtained by noticing that for $D = 0$, equation (96) is an equation of continuity. The equation of characteristic is $dv/dt = -\gamma h(t)v$ which can be integrated into $v(t) = v_0 e^{-\gamma H(t)}$. Writing $P(v,t)dv = P_0(v_0)dv_0$, we finally get equation (116).

If $P_0(v) = \eta(v-v_0)$ is a step function with $\eta(v-v_0) = 1$ for $v < v_0$ and $\eta(v-v_0) = 0$ for $v > v_0$, we find that

$$P(v,t) = H_1(t)\Phi\left(\frac{v-v_0/H_1(t)}{2\chi(t)}\right), \quad (117)$$

where

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_x^{+\infty} e^{-y^2}dy. \quad (118)$$

Alternatively, if $P_0(v) = \delta(v-v_0)$, we obtain

$$P(v,t) = \frac{1}{\sqrt{4\pi\chi(t)^2}}e^{-\frac{(v-v_0/H_1(t))^2}{4\chi(t)^2}}. \quad (119)$$

6.2 The case h(t) = 1

In the ordinary situation where the friction coefficient is constant ($h(t) = 1$), we get

$$H(t) = t, \quad H_1(t) = e^{\gamma t}, \quad (120)$$

$$H_2(t) = \frac{D}{2\gamma}\left(e^{2\gamma t} - 1\right), \quad \chi^2(t) = \frac{D}{2\gamma}\left(1 - e^{-2\gamma t}\right). \quad (121)$$

Equation (117) then takes the form

$$P(v,t) = e^{\gamma t}\Phi\left(\frac{v - v_0 e^{-\gamma t}}{\sqrt{\frac{2D}{\gamma}(1 - e^{-2\gamma t})}}\right), \qquad (122)$$

which behaves like

$$P(v,t) \sim e^{\gamma t}\Phi\left(\frac{v - v_0 e^{-\gamma t}}{\sqrt{2D/\gamma}}\right), \qquad (123)$$

for $t \to +\infty$. On the other hand, equation (119) takes the form

$$P(v,t) = \sqrt{\frac{\gamma}{2\pi D(1 - e^{-2\gamma t})}}e^{-\frac{\gamma(v - v_0 e^{-\gamma t})^2}{2D(1 - e^{-2\gamma t})}}, \quad (124)$$

which tends to the Maxwellian for $t \to +\infty$.

6.3 The case h(t) = t

In the case where the friction coefficient increases linearly with time ($h(t) = t$), we get

$$H(t) = \frac{t^2}{2}, \quad H_1(t) = e^{\gamma\frac{t^2}{2}}, \qquad (125)$$

$$H_2(t) = D\int_0^t e^{\gamma\tau^2}d\tau, \quad \chi^2(t) = De^{-\gamma t^2}\int_0^t e^{\gamma\tau^2}d\tau. \qquad (126)$$

The last function can be written $\chi^2(t) = \frac{D}{\sqrt{\gamma}}\phi(\sqrt{\gamma}t)$ where

$$\phi(t) = e^{-t^2}\int_0^t e^{x^2}dx, \qquad (127)$$

is the Dawson integral. It behaves like $\phi(t) \sim 1/(2t)$ for $t \to +\infty$ and like $\phi(t) \sim t$ for $t \to 0$. Equation (117) then takes the form

$$P(v,t) = e^{\gamma t^2/2}\Phi\left(\frac{v - v_0 e^{-\gamma t^2/2}}{\sqrt{\frac{4D}{\gamma^{1/2}}\phi(\sqrt{\gamma}t)}}\right), \qquad (128)$$

which behaves like

$$P(v,t) \sim e^{\gamma t^2/2}\Phi\left(\frac{\gamma t(v - v_0 e^{-\gamma t^2/2})}{\sqrt{2D}}\right), \qquad (129)$$

for $t \to +\infty$. The time evolution of the profile $P(v,t)$ is represented in Figure 4. On the other hand, equation (119) takes the form

$$P(v,t) = \sqrt{\frac{\gamma^{1/2}}{4\pi D\phi(\sqrt{\gamma}t)}}e^{-\frac{\gamma^{1/2}(v - v_0 e^{-\gamma t^2/2})^2}{4D\phi(\sqrt{\gamma}t)}}. \quad (130)$$

For $t \to +\infty$, it behaves like

$$P(v,t) = \left(\frac{\gamma t}{2\pi D}\right)^{1/2}e^{-\frac{\gamma t}{2D}(v - v_0 e^{-\gamma t^2/2})^2}, \qquad (131)$$
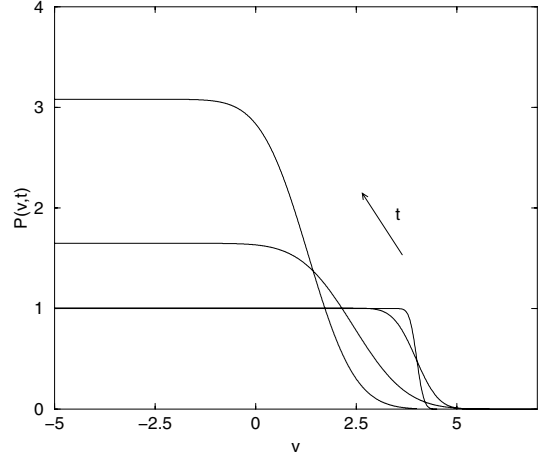

Fig. 4. The function $P(v,t)$ with $\gamma = D = 1$ and $v_0 = 4$ for $t = 0.01, 0.1, 1, 1.5$ starting from a step function $\eta(v - v_0)$.
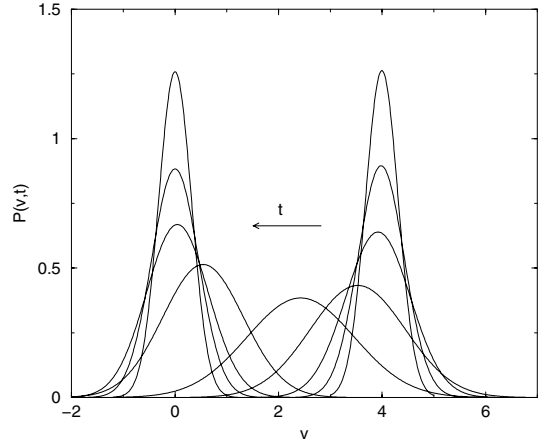

Fig. 5. The function $P(v,t)$ with $\gamma = D = 1$ and $v_0 = 4$ for $t = 0.05, 0.1, 0.2, 0.5, 1, 2, 3, 5, 10$ starting from $\delta(v - v_0)$.

and eventually tends to $\delta(v)$. The system collapses to a Dirac in velocity space due to the divergence of the friction coefficient for $t \to +\infty$. Alternatively, for $t \to 0$, the initial distribution $\delta(v - v_0)$ deforms itself according to

$$P(v,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(v - v_0 e^{-\gamma t^2/2})^2}{4Dt}}. \qquad (132)$$

The time evolution of the profile $P(v,t)$ is represented in Figure 5.

6.4 Another example

For completeness, we also provide the analytical solution of the equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial v}\left(D\frac{\partial P}{\partial v} + \gamma h(t)P\right), \qquad (133)$$

where D and γ are constant parameters. Writing this equation in the form

$$\frac{\partial P}{\partial t} - \gamma h(t)\frac{\partial P}{\partial v} = D\frac{\partial^2 P}{\partial v^2}, \qquad (134)$$

we see that the second term is similar to an advection (in velocity space) by an effective velocity field $V(v,t) = -\gamma h(t)$. Let $v_f(t)$ be a solution of $dv_f/dt = -\gamma h(t)$. We take $v_f(t) = -\gamma H(t)$ where $H(t)$ denotes a primitive of $h(t)$ with $H(0) = 0$. Then, we define $z = v - v_f(t)$ and $P(v,t) = \phi(z,t)$. Substituting these relations in equation (134), we find that $\phi(z,t)$ satisfies the diffusion equation

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial z^2}. \qquad (135)$$

Using for example the results of Section 6.1, the general solution of this equation is

$$\phi(z,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} \phi_0(z - 2\sqrt{Dt}x)dx. \qquad (136)$$

Returning to original variables, we get

$$P(v,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-x^2} P_0(v + \gamma H(t) - 2\sqrt{Dt}x)dx, \qquad (137)$$

where $P_0(v) = P(v,0)$ is the initial value of the probability distribution. If $P_0(v) = \eta(v - v_0)$ is a step function, we find that

$$P(v,t) = \Phi\left(\frac{v - v_0 + \gamma H(t)}{2\sqrt{Dt}}\right). \qquad (138)$$

Alternatively, if $P_0(v) = \delta(v - v_0)$, we obtain

$$P(v,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(v - v_0 + \gamma H(t))^2}{4Dt}}. \qquad (139)$$

7 Conclusion

We have presented first elements of kinetic theory appropriate to inhomogeneous systems with long-range interactions. Explicit results have been obtained for the HMF model in the low temperature limit $T \to 0$ where the particles perform harmonic motion around $\theta = 0$ with the same frequency. These results show that the description of *strongly* inhomogeneous systems is very different from what we are used to in ordinary kinetic theory. In particular, the Kubo formula is not valid in its usual form because the auto-correlation function of the force does not decrease sufficiently rapidly for large times. On the contrary, it has a striking oscillatory behaviour that averages to zero over a period. On the other hand, the Einstein relation is broken. Indeed, the friction coefficient depends on time and diverges linearly as $t \to +\infty$ due to the cumulative effects of friction as the particle undergoes several oscillations.

These results are strikingly different from those obtained in the homogeneous phase $T > T_c$ of the HMF model where the particles follow linear trajectories with constant velocity in a first approximation. In that case, the Kubo formula and the Einstein relation are recovered [7]. It would be interesting to consider the case of the weakly inhomogeneous HMF model. This could be studied perturbatively by expanding the relations (13), (14) close to the critical point $T \to T_c^-$ where $B \to 0$. Similar expansions have been previously considered in [7] in other circumstances. It will be necessary to distinguish between open trajectories that are similar to the situation for $T > T_c$ from closed trajectories that are similar to the situation for $T \to 0$. It would also be interesting to investigate how collective effects can modify these results (as in the case $T > T_c$). In particular, close to the critical point the correlation time diverges which complicates the kinetic theory.

We stress that our approach does not provide a full kinetic theory of the inhomogeneous HMF model. Our main motivation was to mention the difficulties with such a kinetic approach due to Markovian effects and spatial inhomogeneities and to show the differences with more familiar kinetic theories. Even if we do not obtain many explicit predictions, an interesting aspect of our study is to show that the usual kinetic relations (Kubo formula, Einstein relation, ...) can break down for strongly inhomogenous systems with long-range interactions such as the HMF model. Finally, we would like to mention that it is important to develop a kinetic theory for the *inhomogeneous* HMF model because there are situations of physical interest where the system is spatially inhomogeneous. On the other hand, the bath can have a complicated phase-space structure with a hierarchical cluster size distribution as exemplified in the numerical simulations of Rapisarda & Pluchino [34]. The kinetic theory should take into account these effects.

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